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Supersymmetry without Grassmann variables

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ABSTRACT: Supersymmetry transformations may be represented by unitary operators in a formulation of supersymmetry without numbers that anti-commute. The physical relevance of this formulation hinges on whether or not one may add states of even and odd fermion number, a question which soon may be settled by experiment.

KEYWORDS: Space-Time Symmetries, Global Symmetries, Extended Supersymmetry.

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1. Introduction

Supersymmetry usually is formulated in terms of anti-commuting numbers called Grassmann variables. In these formulations supersymmetry transformations $T(\xi)$ are represented by exponentials $T(\xi) = \exp(\xi Q + \bar{Q}\bar{\xi})$ of the supercharges Q_a multiplied by the components ξ^a of a Grassmann spinor [1]. But since the Hilbert space $\mathcal{H}$ of quantum mechanics is a vector space over the complex numbers, and not over a field containing Grassmann variables, the image $T(\xi)|\psi\rangle$ of any state $|\psi\rangle$ not annihilated by the fermionic generators Q_a will lie outside of $\mathcal{H}$. One may include states like $T(\xi)|\psi\rangle$ either by enlarging the Hilbert space $\mathcal{H}$ or by expressing the transformations of supersymmetry as unitary transformations on $\mathcal{H}$. The latter course is the subject of this article.

Inasmuch as supersymmetric quantum field theories are merely ordinary quantum field theories with particular fields and parameters, it is clear that one can discuss supersymmetry without the use of Grassmann variables. Moreover if z^a is a complex spinor, then the operator $G(z) = zQ + \bar{Q}\bar{z}$ is hermitian, $G(z)^\dagger = G(z)$, and the operator $U(z) = \exp(-iG(z))$ is unitary, $U(z)^\dagger = U(z)^{-1}$. But the image $U(z)|\psi\rangle$ of any state $|\psi\rangle$ not annihilated by the fermionic generators Q_a must be a superposition of states of even and odd fermion number. It is generally believed, however, that such states are unphysical because under a rotation of 2π around any axis, they change by more than an overall phase [2]. If this superselection rule is true, then the state $U(z)|\psi\rangle$ lies outside of the physical Hilbert space $\mathcal{H}$, and we gain very little by considering the unitary operator $U(z)$.

But this superselection rule is true or false according to whether the Lorentz group of nature is $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ or $\mathrm{SL}(2,\mathbb{C})$, and its validity can be settled only by experiment because the consequences of $\mathrm{SL}(2,\mathbb{C})$ are the same as those of $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ except for the absence of superselection rules [3]. Andreev [4] has suggested how this rule may be tested by experiments on mesoscopic metal particles at low temperatures. Efforts to perform such experiments are in progress.

It may make sense therefore to resolve some of the technical problems that arise when one tries to represent the transformations of supersymmetry by unitary operators. The case of the chiral multiplet will be considered in this paper.

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2. Unitary supersymmetry transformations

In what follows it will be shown how to implement supersymmetry transformations on the chiral multiplet by means of unitary operators without Grassmann variables. The supersymmetry transformations generated by the operator $\mathcal{G}(\xi) = \xi Q + \bar{Q}\bar{\xi}$, in which ξ^a is a Grassmann variable and Q_a a supercharge, will be recalled, and the transformations induced by the generator $G(z) = zQ + \bar{Q}\bar{z}$, where z is a complex number, will be discussed. It will be shown that these transformations are a symmetry of the theory because the change in the action density is a divergence of a current. The Noether procedure then will be used to show that the fermionic generators of this symmetry are the usual supercharges Q_a, which satisfy the algebra of supersymmetry. Thus the unitary operator $U(z) = \exp[-iG(z)]$ represents a supersymmetry transformation.

The action density for the chiral multiplet with superpotential $W(A)$ is

$$\mathcal{L} = \frac{i}{2}\partial_n\bar{\psi}\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\partial_n\psi - \partial_n\bar{A}\partial^n A - |W'|^2 - \frac{1}{2}W''\psi\psi - \frac{1}{2}\bar{W}''\bar{\psi}\bar{\psi} \tag{2.1}$$

in which $W' = \partial W(A)/\partial A$. The supercharges Q_a are

$$Q_a = \sqrt{2}\int d^3x\, \left(\sigma^m_{a\dot{b}}\bar{\sigma}^{0\dot{b}c}\psi_c\partial_m\bar{A} - i\sigma^0_{a\dot{b}}\bar{\psi}^{\dot{b}}\bar{W}'\right). \tag{2.2}$$

In a mixed notation they are

$$\begin{aligned} Q_a &= -\sqrt{2}\int d^3x\, \left(\sigma^m_{a\dot{b}}\psi_b\partial_m\bar{A} - i\bar{\psi}^{\dot{a}}\bar{W}'\right) & (2.3)\\ &= -\sqrt{2}\int d^3x\, \left(\sigma^m_{a\dot{b}}\psi_b\partial_m\bar{A} - i\varepsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}}\bar{W}'\right) & (2.4)\end{aligned}$$

or more simply

$$Q = \sqrt{2}\int d^3x\, \left[\left(\partial_0\bar{A} - \vec{\sigma}\cdot\nabla\bar{A}\right)\psi - \bar{W}'\,\sigma^2\bar{\psi}\right]. \tag{2.5}$$

In the usual formalism with Grassmann variables ξ^a, the "bosonic" operator

$$\mathcal{G}(\xi) = \xi Q + \bar{Q}\bar{\xi} \tag{2.6}$$

generates in the fields $A(x)$ and $\psi(x)$ the changes

$$\begin{aligned} \delta_\xi A(x) \equiv [i\mathcal{G}(\xi), A(x)] &= i\sqrt{2}\int d^3y\,\xi^a\sigma^m_{a\dot{b}}\psi_b(y)\left[A(x),\partial_m\bar{A}(y)\right]\\ &= -\sqrt{2}\int d^3y\,\xi^a\sigma^0_{a\dot{b}}\psi_b(y)\delta(\vec{x}-\vec{y})\\ &= \sqrt{2}\xi^a\psi_a(x) = \sqrt{2}\xi\psi\,, \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
\delta_\xi \psi_a(x) &\equiv [i\mathcal{G}(\xi), \psi_a(x)] \\
&= \sqrt{2}\int d^3y \left[\psi_a(x), \xi^b \varepsilon^{\dot{b}\dot{c}}\bar{\psi}_{\dot{c}}(y)\bar{W}' + i\bar{\psi}_{\dot{c}}(y)\sigma^m_{c\dot{b}}\bar{\xi}^{\dot{b}}\partial_m A(y)\right] \\
&= \sqrt{2}\left(-\xi^b\varepsilon^{\dot{b}\dot{a}}\bar{W}' + i\sigma^m_{a\dot{b}}\bar{\xi}^{\dot{b}}\partial_m A(x)\right) \\
&= \sqrt{2}\left(\xi^b\varepsilon_{ba}\bar{W}' + i\sigma^m_{a\dot{b}}\bar{\xi}^{\dot{b}}\partial_m A(x)\right) \\
&= \sqrt{2}i\sigma^m_{a\dot{b}}\bar{\xi}^{\dot{b}}\partial_m A(x) - \sqrt{2}\xi_a\bar{W}' ,
\end{aligned}
\tag{2.8}
$$

which is the usual result if we use $-\bar{W}' = F$, where F is the auxiliary field.

Exponentials of the generator $\mathcal{G}(\xi) = \xi Q + \bar{Q}\bar{\xi}$ are not unitary operators because they involve Grassmann variables. Can one avoid these anti-commuting variables? Let us consider using generators $G(z)$ that are complex linear forms in the supercharges Q and $\bar{Q}$

$$G(z) = zQ + \bar{Q}\bar{z}\,, \tag{2.9}$$

where z^a is a complex spinor. Now the change in the field $A(x)$ is

$$
\begin{aligned}
dA(x) \equiv [\, iG(z), A(x)\,] &= -\,i\sqrt{2}\int d^3y\, z^a\sigma^m_{a\dot{b}}\psi_b(y)\,[\,\partial_m\bar{A}(y), A(x)\,] \\
&= -\,\sqrt{2}\int d^3y\, z^a\sigma^0_{a\dot{b}}\psi_b(y)\delta(\vec{x}-\vec{y}) \\
&= \sqrt{2}z^a\psi_a(x) = \sqrt{2}z^a\psi_a\,,
\end{aligned}
\tag{2.10}
$$

which is the same as (2.7) except that the Grassmann spinor ξ has been replaced by the complex spinor z. The conjugate change is

$$d\bar{A} = \sqrt{2}\bar{\psi}_{\dot{a}}\bar{z}^{\dot{a}}\,. \tag{2.11}$$

This procedure will not work, however, for the Fermi field ψ. Instead we must write $d\psi$ as an anti-commutator. There are several ways of doing this, but if we want $d\psi$ to be the adjoint of $d\bar{\psi}$, then we can not have a single rule for the change in the product of two spinor fields irrespective of whether they transform like ψ or like $\bar{\chi}$. We choose to have $d\psi$ be the adjoint of $d\bar{\psi}$, and so we shall have four different rules for the change in the product of two spinor fields. We define

$$
\begin{aligned}
d\psi_a(x) &\equiv -\,\{iG(z), \psi_a(x)\} \\
&= \sqrt{2}\int d^3y \left\{z^b\varepsilon^{\dot{b}\dot{c}}\bar{\psi}_{\dot{c}}(y)\bar{W}' + i\bar{\psi}_{\dot{c}}(y)\sigma^m_{c\dot{b}}\bar{z}^{\dot{b}}\partial_m A(y), \psi_a(x)\right\} \\
&= \sqrt{2}\left(z^b\varepsilon^{\dot{b}\dot{a}}\bar{W}' + i\sigma^m_{a\dot{b}}\bar{z}^{\dot{b}}\partial_m A(x)\right) \\
&= \sqrt{2}\left(-z^b\varepsilon_{ba}\bar{W}' + i\sigma^m_{a\dot{b}}\bar{z}^{\dot{b}}\partial_m A(x)\right) \\
&= \sqrt{2}i\sigma^m_{a\dot{b}}\bar{z}^{\dot{b}}\partial_m A(x) + \sqrt{2}z_a\bar{W}'\,.
\end{aligned}
\tag{2.12}
$$

The change in the conjugate $\bar\psi$ is the conjugate of the change in ψ

$$\begin{aligned} d\bar\psi_{\dot a} &\equiv \left\{iG(z), \bar\psi_{\dot a}\right\} = (-\left\{iG(z), \psi_a(x)\right\})^\dagger = (d\psi_a)^\dagger \\ &= -\sqrt{2} i z^b \sigma^m_{b\dot a} \partial_m \bar A + \sqrt{2} \bar z_{\dot a} W' . \end{aligned} \tag{2.13}$$

Although these formulas differ from expression (2.8) for $\delta\psi$ and its conjugate for $\delta\bar\psi$ by the signs of their second terms, and of course by the replacement of a Grassmann spinor ξ by a complex one z, we shall see that these sign differences are appropriate and that supersymmetry can be implemented by unitary transformations acting on the states and physical operators of the theory.

The key point is that the physical operators of the theory contain even powers of the Fermi fields. Thus the change in the generic product $\psi\chi$ of two Fermi fields is

$$\begin{aligned} d(\psi\chi) &= [\, iG(z), \psi\chi\,] \\ &= iG(z)\,\psi\chi - \psi\chi\, iG(z) \\ &= iG(z)\,\psi\chi + \psi\, iG(z)\,\chi - \psi\, iG(z)\,\chi - \psi\chi\, iG(z) \\ &= \{iG(z), \psi\}\,\chi - \psi\,\{iG(z), \chi\} \\ &= -\,d\psi\,\chi + \psi\, d\chi\,, \end{aligned} \tag{2.14}$$

in which the spinor indices, which may be different, are suppressed. It is easy to see that the other three rules are:

$$d(\bar\psi\bar\chi) = \left[\, iG(z), \bar\psi\bar\chi\,\right] = d\bar\psi\,\bar\chi - \bar\psi\, d\bar\chi \tag{2.15}$$

$$d(\bar\psi\chi) = \left[\, iG(z), \bar\psi\chi\,\right] = d\bar\psi\,\chi + \bar\psi\, d\chi \tag{2.16}$$

$$d(\psi\bar\chi) = [\, iG(z), \psi\bar\chi\,] = -d\psi\,\bar\chi - \psi\, d\bar\chi\,. \tag{2.17}$$

Let us now consider the effect of these transformations on the chiral action density (2.1). The change in $\mathcal{L}$ due to the changes dA and $d\psi$ and their conjugates is

$$\begin{aligned} d\mathcal{L} = [\, iG(z), \mathcal{L}\,] = {} & \frac{i}{2}\partial_n d\bar\psi\bar\sigma^n\psi + \frac{i}{2}\partial_n\bar\psi\bar\sigma^n d\psi - \frac{i}{2} d\bar\psi\bar\sigma^n\partial_n\psi - \frac{i}{2}\bar\psi\bar\sigma^n\partial_n d\psi - \\ & -\partial_n d\bar A\partial^n A - \partial_n\bar A\partial^n dA - \bar W' W'' dA - W'\bar W'' d\bar A - \\ & -\frac{1}{2} W''' dA\psi\psi - \frac{1}{2}\bar W''' d\bar A\bar\psi\bar\psi + \frac{1}{2} W'' d\psi\psi - \frac{1}{2} W''\psi d\psi - \\ & -\frac{1}{2}\bar W'' d\bar\psi\bar\psi + \frac{1}{2}\bar W''\bar\psi d\bar\psi\,. \end{aligned} \tag{2.18}$$

The part of $d\mathcal{L}$ that depends upon z is

$$\begin{aligned} d_z\mathcal{L} = {} & \frac{i}{2}\partial_n(-\sqrt{2} i z\sigma^m\partial_m\bar A)\bar\sigma^n\psi + \frac{i}{2}\partial_n\bar\psi\bar\sigma^n(\sqrt{2} z\bar W') - \frac{i}{2}(-\sqrt{2} i z\sigma^m\partial_m\bar A)\bar\sigma^n\partial_n\psi - \\ & -\frac{i}{2}\bar\psi\bar\sigma^n\partial_n(\sqrt{2} z\bar W') - \partial_n\bar A\partial^n(\sqrt{2} z\psi) - \bar W' W''(\sqrt{2} z\psi) - \frac{1}{2} W'''(\sqrt{2} z\psi)\psi\psi + \\ & +\frac{1}{2} W''(\sqrt{2} z\bar W')\psi - \frac{1}{2} W''\psi(\sqrt{2} z\bar W') - \frac{1}{2}\bar W''(-\sqrt{2} i z\sigma^m\partial_m\bar A)\bar\psi + \\ & +\frac{1}{2}\bar W''\bar\psi(-\sqrt{2} i z\sigma^m\partial_m\bar A). \end{aligned} \tag{2.19}$$

Because Fermi fields at equal times anti-commute, the term proportional to W''' vanishes. The two terms proportional to $\bar{W}'\,W''$ cancel. The last two terms may be written as

$$\frac{i}{\sqrt{2}}\bar{\psi}^{\dot{a}}\sigma^m_{b\dot{a}}z^b\partial_m\bar{W}' = \frac{i}{\sqrt{2}}\bar{\psi}_{\dot{c}}\,\varepsilon^{\dot{a}\dot{c}}\sigma^m_{b\dot{a}}\varepsilon^{bd}z_d\,\partial_m\bar{W}' = \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^m z\,\partial_m\bar{W}' \tag{2.20}$$

and as

$$-\frac{i}{\sqrt{2}}\bar{\psi}_{\dot{a}}z^b\sigma^m_{b\dot{c}}\,\varepsilon^{\dot{a}\dot{c}}\partial_m\bar{W}' = -\frac{i}{\sqrt{2}}\bar{\psi}_{\dot{a}}\varepsilon^{\dot{a}\dot{c}}\varepsilon^{bd}\sigma^m_{b\dot{c}}z_d\,\partial_m\bar{W}' = \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^m z\,\partial_m\bar{W}'\,. \tag{2.21}$$

So the change $d_z\mathcal{L}$ in the action density is

$$d_z\mathcal{L} = \frac{1}{\sqrt{2}}z\sigma^m\bar{\sigma}^n\psi\partial_n\partial_m\bar{A} - \frac{1}{\sqrt{2}}z\sigma^m\bar{\sigma}^n\partial_n\psi\partial_m\bar{A} - \sqrt{2}z\partial^n\psi\partial_n\bar{A} + \frac{i}{\sqrt{2}}\partial_n\left(\bar{\psi}\bar{\sigma}^n z\bar{W}'\right). \tag{2.22}$$

We may write this change $d_z\mathcal{L}$ as the total divergence

$$d_z\mathcal{L} = \partial_n K^n_z \tag{2.23}$$

of the current

$$K^n_z = -\frac{1}{\sqrt{2}}z\sigma^m\bar{\sigma}^n\partial_m\bar{A} - \sqrt{2}z\psi\partial^n\bar{A} + \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n z\bar{W}'\,, \tag{2.24}$$

which shows that the action is invariant under the unitary transformation

$$U(z) = e^{-iG(z)} \tag{2.25}$$

at least for infinitesimal values of the complex spinor z.

The Noether current associated with the susy transformation (2.10)–(2.13) of the action density (2.1) is

$$\begin{aligned} J^n &= \frac{i}{2}d\bar{\psi}\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n d\psi - d\bar{A}\partial^n A - \partial^n\bar{A}dA \\ &= \frac{i}{2}\left(-i\sqrt{2}z\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{z}W'\right)\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\left(i\sqrt{2}\sigma^m\bar{z}\partial_m A + \sqrt{2}z\bar{W}'\right) - \\ &\quad -\sqrt{2}\bar{z}\bar{\psi}\partial^n A - \partial^n\bar{A}\sqrt{2}z\psi\,. \end{aligned} \tag{2.26}$$

The part depending on z is

$$J^n_z = \frac{1}{\sqrt{2}}z\sigma^m\bar{\sigma}^n\psi\partial_m\bar{A} - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n z\bar{W}' - \sqrt{2}z\psi\partial^n\bar{A}\,. \tag{2.27}$$

The Noether current J^n satisfies

$$d\mathcal{L} = \partial_n J^n\,, \tag{2.28}$$

and so the difference $J^n - K^n$ of the two currents

$$S^n = J^n - K^n \tag{2.29}$$

is conserved

$$\partial_n S^n = 0\,. \tag{2.30}$$

The current S^n is the supercurrent of the Lagrange density $\mathcal{L}$. The part S^n_z that depends upon z is simply

$$S^n_z = \sqrt{2} z \sigma^m \bar\sigma^n \psi \partial_m \bar A - i\sqrt{2}\bar\psi\bar\sigma^n z \bar W'\,. \tag{2.31}$$

Thus the quantity zQ is

$$zQ = \int d^3x\, S^0_z = \int d^3x\, \sqrt{2} z \sigma^m \bar\sigma^0 \psi \partial_m \bar A - i\sqrt{2}\bar\psi\bar\sigma^0 z \bar W'\,, \tag{2.32}$$

and so by the identity $\bar\psi\bar\sigma^0 z = z\sigma^0\bar\psi$, the supercharges Q_a are

$$Q_a = \sqrt{2}\int d^3x\, \left(\sigma^m_{ab}\bar\sigma^{0bc}\psi_c\partial_m\bar A - i\sigma^0_{ab}\bar\psi^b\bar W'\right). \tag{2.33}$$

These supercharges, which generate the unitary supersymmetry transformations (2.25) and (2.10)–(2.13), are the same as the those (2.2) that generate the Grassmann supersymmetry transformations (2.7) and (2.8).

We have seen that the action density (2.1) is invariant under the unitary transformation (2.25) which is an exponential of an imaginary linear form in the supercharges (2.2), that the induced complex supersymmetry transformations (2.10)–(2.13) differ somewhat from the usual supersymmetry transformations (2.7) and (2.8), and that the supercharges derived by the Noether technique (2.18)–(2.33) from the complex supersymmetry transformations (2.10)–(2.13) are the same as the conventional supercharges (2.2).

It is straightforward to generalize this argument to the general chiral theory consisting of N multiplets A_i, ψ_i, F_i interacting through an arbitrary analytic superpotential $W(A_1, \ldots, A_N)$. In fact when appropriately generalized, the rules (2.9)–(2.17) should apply to any supersymmetric field theory. Unitary operators without Grassmann variables can implement supersymmetry transformations upon the action and upon other operators that involve even powers of Fermi fields.

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